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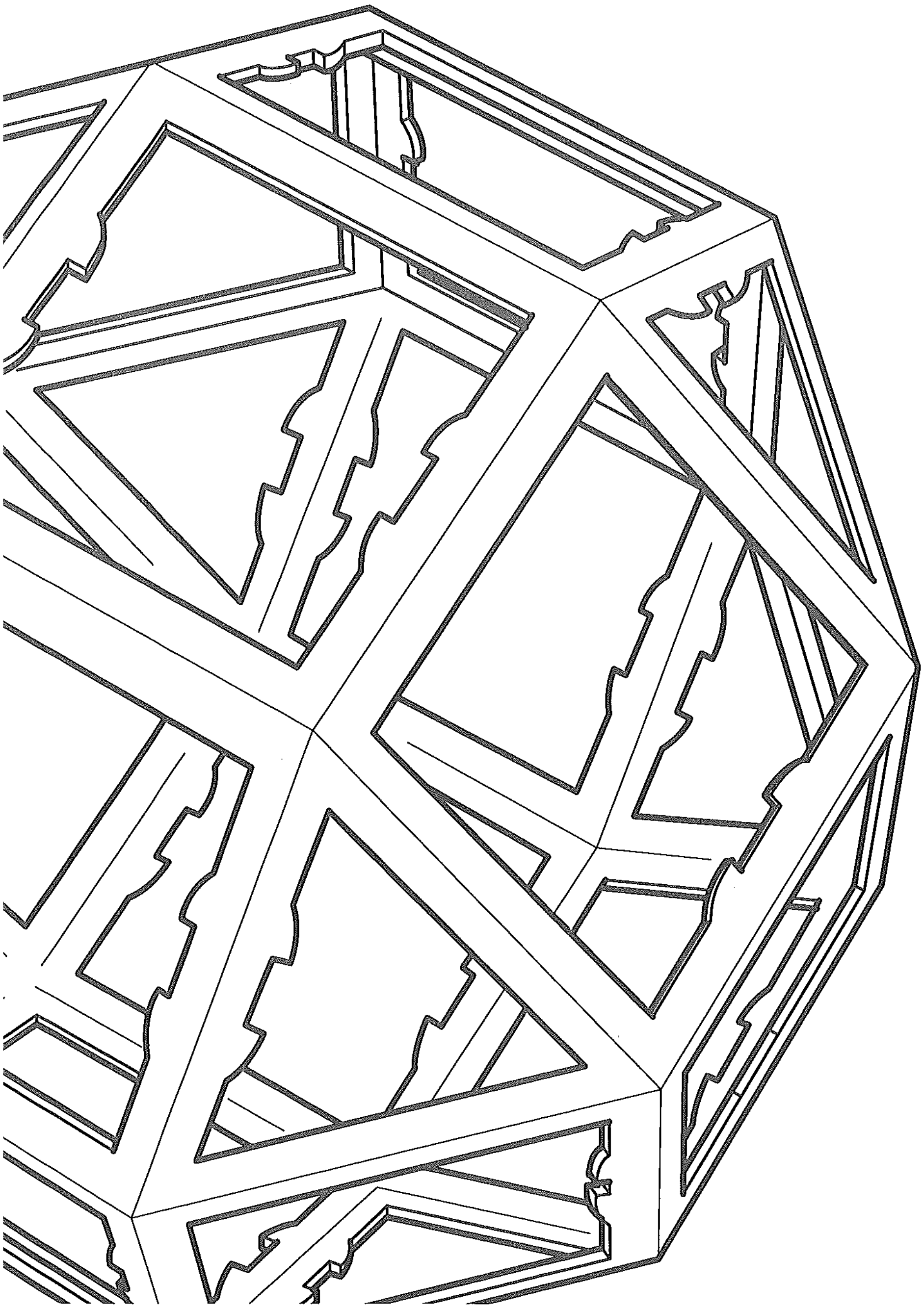
The Nature of Innovation and Advancement in Mathematics:

A Philosophical Investigation

by James Davies



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The Nature of Innovation and Advancement in Mathematics: a Philosophical Investigation

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Abstract

This paper seeks a philosophical explanation as to why certain mathematical settings seem more accomodating to certain theorems, especially when this is a new setting formulated after the theorem. I will emphasise the distinction between the formal language of mathematical symbolism and its informal counterpart, conceived as a metalangauge. This distinction is characterised using the intension/extension distinction, borrowed from the study of natural languages. The same natural language distinction is then applied to the notion of logical consequence, in order to clarify the conceptual relationships between old and new settings. It will turn out that a semantic conception of mathematical truth best explains the foregoing conclusions while at the same time avoiding the pitfalls of traditional Platonism.

1 Introduction

In the history of mathematics, there are many instances of propositions appearing resistant to proof for long periods of time, until a substantial reconceptualisation take place. At this point, such difficult theorems can take on a new, more tractable form. For instance, Riemann saw that elliptic functions exhibit more comprehensible behaviour when defined over a 'doughnut' of complex values, as opposed to the complex plane.[1] In terms of the progress of mathematics as a science, this is a good thing. The mathematician Philip Davis notes how

> ...in mathematics there is a long and vitally important record of impossibilities being broken by the introduction of structural changes, (1987; cited in Wilson [1992], p.150)

However, from a philosophical point of view, this 'vitally important' phenomenon raises a problem. Mathematical truths are usually taken to be eternally true and, as such, are independant of events which occur in the physical world. When a reconceptualisation takes place, it is often the case that while some statements in the original context benefit from the change, others that were true (maybe trivially so) will become false; for instance, when moving from an affine geometrical setting to a projective one, the statement that parallel lines never meet is falsified. Thus, the idea that mathematical impossibilities can be 'broken' challenges the supposed permanence of mathematical knowledge.

In this discussion, I will focus on one aspect of this 'breaking of impossibilities'. The 'introduction of structural changes' will only *fruitfully* break impossibilities if such changes result in what may be termed a more 'natural' conception of the problems which led to the need for the changes in the first place. Thus, in §1 I will give an example of a structural reconceptualisation rendering possible new mathematical results. §2 will begin a philosophical analysis of this phenomenon, beginning with a treatment of the relation between the roles played by deductive rigour and conceptual clarity in constituting mathematical knowledge. The notion of logical consequence will be examined in §3, for if such structural reconceptu-

[1] Wilson (1992), p.151

alisations are fruitful, then it would appear that there is some kind of logical relationship between the old setting and the new. §4 will sketch a philosophical view of the nature of mathematical knowledge which can incorporate, and account for, the conclusions of §§2 and 3.

2 An Example of Reconceptualisation

Consider the following figure on the Euclidean plane: The lines AB and BC are both tangent

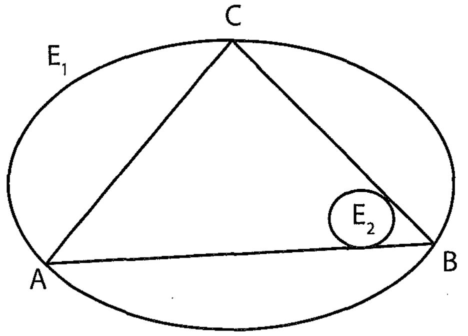

Figure 1

to the ellipse E_1; however, the line CA clearly is not. This leads to the conclusion that "...for two ellipses, one within the other, there is usually no triangle inscribed in one of the ellipses circumscribed about the other."[2] However, an exception is when the ellipses are actually circles,[3] and the triangle is equilateral (figure 2). Nevertheless, it can be shown that this

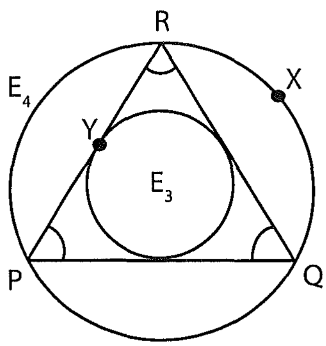

Figure 2

[2]Rosenbaum (1963), p.10 (pp. 10-17 of this volume is my source for this scetion)

[3]While it is true that from an affine point of view, there is no difference between a circle and an ellipse, I will use the term 'circle' in its everyday meaning to improve clarity

is not a singular exception. Consider the act of 'projecting' from a point Q through the plane which contains figure 2 onto another, non-parallel plane (figure 3): the points X and

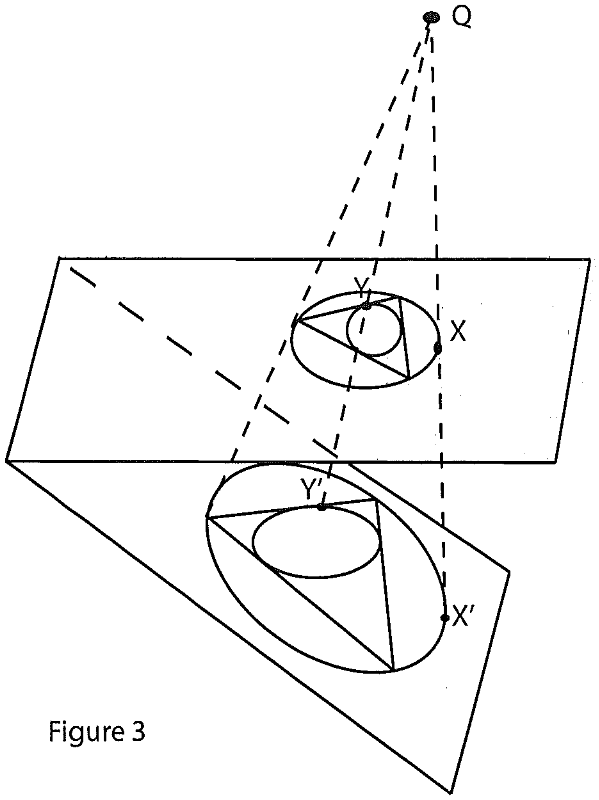

Figure 3

Y on the original plane have been mapped to the points X' and Y' on the new plane by the projection from Q (as have all the other points which lie on the original plane). Thus, the new plane contains an object which looks like figure 4: we have two ellipses E_3' and E_4' which

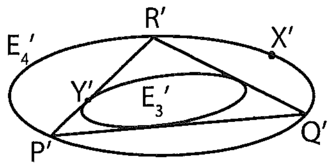

Figure 4

are definitely not circles, and the triangle $P'Q'R'$ is not equilateral. However, the lines of the triangle are all tangential to the inner ellipse. Because figure 2 is symmetrical, the vertices of the triangle could be any set of three points on the outer circle which are equidistant from each other along E_4 - and the result of projection from Q as in figure 3 would yield a shape on the new plane different from that in figure 4. Nevertheless, it will still be the case that the three sides of the new triangle are all tangential to the inner ellipse. This is a particular case of a theorem due to Poncelet:

> If two conics C_1 and C_2 are such that an n-gon can be inscribed in C_1 and circumscribed about C_2, then there are infinitely many n-gons in the same relationship to the conics, with any point on C_1 serving as a vertex of an n-gon. (Rosenbaum (1963), p.12)

However, this argument from projection requires the assumption, implicit in the discussion above, that projection is bijective - that is, all points in the original plane are projected to one and only one point on the new plane. Things standing as they are, this can only be the case when the planes are parallel. If they are not (as in figure 3), then any location of the origin of projection will result in at least one projective line failing to meet the new plane (figure 5). This happens when the line produced by the projection from Z through P is *parallel* to the new plane. Thus, the operation of projection is not in general a one-to-one correspondence. This results in the loss of general certainty regarding whether a point which is a vertex of the inscribed n-gon tangential to the inner ellipse will be projected to a point on the new plane with the same properties[4] - or to a point at all. In order to eliminate the exceptional case where the projected line from Z through P is parallel to the new plane, projective geometers add a so-called 'ideal point', which is defined to be the point at which the line from Z through P meets the plane it is parallel to. If one imagines standing on a set of railway tracks, looking down the parallel toward the horizon, then one observes that the two tracks appear to meet at the horizon - an 'infinitely far' distance (no matter how long one spends walking the tracks,

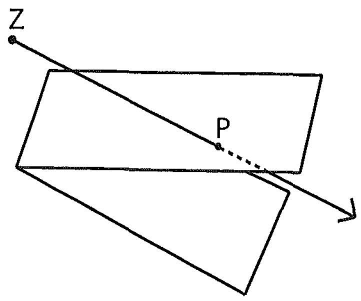

Figure 5

[4]Disregarding metric properties, which are not preserved under projection.

the point at which they (appear to) intersect will never be reached). Thus the term 'point at infinity' which is used to describe the points at which parallels meet, and 'line at infinity' for the horizon (the line comprised of all the 'points at infinity'). It will be worth noting that there is only ever one point at infinity for each direction, no matter how many parallel lines are running in that direction - just as if one is standing on a large rail-line with many parallel tracks running in the same direction, they still appear to meet at a single point.

Thus, it seems that adding points which violate Euclid's parallel postulate - i.e. replacing 'two distinct lines in a plane either have one and only one point in common or are parallel' with 'two distinct lines in a plane have one and only one point in common'[5] results in what many have termed a 'completion' of Euclidean geometry.[6] Once equipped with ideal points and the line at infinity, geometers were able to show that not only were many difficult theorems of Euclidean geometry were made more tractable in the new 'setting' (the one with points at infinity), such as Poncelet's theorem, above, but also that the projective setting (combined with complex co-ordinate points) enabled analytic proofs of situations which appear totally distinct from the synthetic point of view[7]. New discoveries were occurring due to the addition and acceptance of seemingly obscure objects (the term 'imaginary' for the complex number $i = \sqrt{-1}$) was originally intended to be derogetory). It is this phenomenon, of a 'change of setting' - a reconceptualisation - resulting in the 'conceptual clarification' which is displayed by the increased ease of solving old conjectures and the appearance of new theorems, which I will seek to philosophically address.

3 Change of Setting and 'Naturalness'

The philosopher and mathematician, René Descartes, considered mathematical knowledge to be indubitable and a priori because it is possessed of two defining qualities: clarity and distinctness. Mathematical objects are *distinct* because they are defined by all and only those statements which are true of them. For instance, the number '7' is defined by all the sums,

[5]ibid, p.16

[6]"The new elements are added to complete the mechanism that makes Euclidean geometry work." (Wilson (1992), p.161)

[7]see Wilson (1992) for an example

differences, products, etc. which it produces when combined with other numbers.[8] This is in direct contrast to scientific objects (such as organisms, molecules, and planets) which are considered to be essentially constant even when statements formerly thought true of them are falsified (light is not considered to be a physically different phenomenon to what it was in the 17th century, even though we now know that it does not always travel in 'straight' lines). Mathematical knowledge is *clear* because it provides conceptual reification of whatever is being studied; for instance, the proof of the Pythagorean theorem adds to our concept of 'triangle' by making explicit general facts about certain types of triangles which implicitly follow from what it is for those triangles to be those types of triangles in the first place.

Kenneth Manders, in his (1987), argues that the modern approach to the analysis of mathematical knowledge (via mathematical logic) has singularly addressed the epistemological issue of *reliability*, in terms of analysing the methods of inference employed in mathematical reasoning. Under this conception, the hallmark of mathematical knowledge is the iron-clad reliability of its inferences; and it is the critical analysis of these inferences which provides the research programme for mathematical logic. In terms of Descartes' distinction between clarity and distinctness, reliability is aligned with distinctness, as "...the aspect of distinctness [makes] sure that the objects of study are precisely determined and reliably reasoned about."[9]

One issue Manders raises with this conception is that it lends itself to taking individual propositions[10] as the units of investigation. There is little scope for addressing mathematical theories[11] as wholes, and thus likewise for investigating the conceptual relationships between theories, especially in terms of why it is that some theories seem to be more accomodating to certain theorems than others. From the reliability-centred point of view, "...mathematical progress is nothing but piling up one theorem on another."[12]

[8]It is true that this seems to not be that case when it comes to transfinite numbers: $7 \cdot \aleph_0 = \aleph_0$, as does $n \cdot \aleph_0$ for any $n \leq \aleph_0$; however this can be taken as a defining characteristic of what it means to be 'a number less than or equal to $\aleph_0$

[9]Manders (1987), p.202

[10]and inferences: an inference from A to B can be rendered as the proposition 'if A, then B', or 'from A, infer B'.

[11]Throughout this discussion, I will be talking of mathematical of mathematical 'systems' and mathematical 'settings'. While these terms are not intended to be synonymous, they overlap considerably; in terms of the discussion in §§3 and 4, a setting is 'rigourised' into a system, through the introduction of deductively closed codifications of the setting into a formal language. The term 'theory' is intended to cover both cases.

[12]Manders (1987), p.195

In opposition to this kind of approach, Manders advocates a return to the Cartesian emphasis on the *interdependance* of clarity and distinctness in characterising mathematical knowledge. As the conceptual clarity of a theorem is a product of the relationship between the theorem and its background theory, it is a *global* feature of the background theory, rendering the distinctness/reliability approach inadequate due to its emphasis on individual propositions as autonomous objects of knowledge.[13] This is doubly so when seeking to understand why some theories render some theorems more comprehensible (or 'natural'), as this involves a higher-level relationship between the theory where the theorem or conjecture was originally formulated, and the new theory which renders it further comprehensible. Hence, I will seek an explication of this notion of the clarity component of 'naturalness', by presenting it in contrast with distinctness/reliability. I will argue that seeing distinctness and clarity as working *in tandem* yields a way of viewing the conceptual relationships between mathematical theories which can account for the phenomenon of 'naturalness'.

Let us begin with the distinctness/reliability condition. In modern mathematics, the ultimate standard of reliability is the (possibility, in principle of) embedding the system into set theory. Under this conception, all mathematical objects are sets, and all operations are operations on sets; the languages of individual theories are just convenient shorthands for the language of sets. However mathematical objects are entirely constituted by what is *true* of them. There is nothing more to any number than its logical interactions with other numbers - and thus, nothing more to any set than its logical interactions with other sets. From a logical point of view, the result of this conception of mathematics there is *no way to conceive* of there being any conceptual continuity between two different mathematical theories. For instance, in the system of natural numbers, some differences cannot be calculated due to the absence of negative numbers. Thus

$$\forall x \forall y \exists z (x - y = z)$$

is false when x,y,z are arbitrarily taken from $\mathbf{N}$, but true when taken from $\mathbf{Z}$. Hence, by the converse of Leibniz's identity of indiscernibles[14] these x,y,z denote different objects when those

[13]ibid

[14]If everything true of an object A is true of an object B, then $A = B$. The converse would be, if at least one thing true of A is not true of B, then $A \neq B$.

objects are taken from **Z** as opposed to **N** and thus the above logical sentence means different things. Hence '5' and '+5' denote different objects. These considerations can be extrapolated to apply not just to numbers, but also to functions, sets, and all other mathematical objects. Thus, while objects *internal* to a mathematical theory are interdependant in terms of their properties and relations (if one property or relation is altered, the whole theory undergoes a systematic change), the total opposite is the case when it comes to an inter-theory perspective.

The outcome of this, as mentioned above, is the complete loss of the ability to even conceive of any conceptual continuity between different settings. It may be countered that this is where set theory is able to do some explaining: if two theories are both able to be embedded in set theory, and thus all objects of both theories are treated as sets and all operations on those objects as sets, then we have a way to conceive of continuity between theories. '5' and '+5' both denote the same set.[15] However, while this does supply continuity, it completely does away with the other aspect of of our investigation - that the formulation of new settings which leads to increased comprehensibility of theorems and conjectures is a non-trivial intellectual accomplishment. In Manders' words,

> [S]et theoretic definability [does not] *by itself* set apart those relationships between (set theoretically definable) conceptual settings which constitute successful reconceptualisations...from the infinitely many completely uninteresting ones. (1987, p.200; emphasis added)

Thus, we have a dichotomy. On the one hand, one setting cannot be a more natural home for a theorem than another because the expression in the new setting expresses something different. Elliptic functions do not achieve additional clarity when defined over a complex doughnut rather than the complex plane, because these are definitionally different functions. On the other hand, one setting cannot be more natural than another because *all* settings are interpretations of set theory, and thus enjoy the same epistemic status.

Over-emphasis on the reliability-theoretic approach also results in the loss of our other

[15] up to isomorphism: e.g. Zermelo vs. von Neumann ordinals.

major concern, *clarity*. Manders notes that "...fully formalised proofs are often unintelligible...increased precision is often achieved at the expense of clarity."[16] Likewise, Saunders Mac Lane observes that

> All mathematics can indeed be built up within set theory, but the description of many mathematical objects as structures [i.e., as sensitive to setting] is much more illuminating then some explicit set-theoretic description. (1996, p.182)

To state the obvious, most mathematicians do not 'think' in set theory. They think in what may be termed an informal metalangauge, where intuitive notions are used and explored in finding inspiration for taking research in certain directions, or using a certain proof strategy in some particular instance. If the only fundamental task in the philosophy of mathematics were to formulate a logical system which can display the reliability of mathematical knowledge, then statements such as those above would not be finding their way into published papers.

I will now show that it can be seen that the reliability approach, and its attendant loss of conceptual continuity and/or respect for innovation, are the result of an implicit assumption that mathematics is purely what is termed an *extensional* matter. To clarify this claim, i will give a brief outline of the extension/intension distinction.

In natural language, most terms have both an extension and an intension. The extension is the collection of objects to which the term applies; the intension is what it is that makes those objects part of that collection. For (a well-worn) example, the terms 'creature with a heart' and 'creature with a kidney' have the same extension (on planet Earth, at least). However, this is the result of contingent facts about the way organic life has evolved on our planet; there is no necessity in the answer as to *why* the extensions are the same. In the formal study of the semantics of natural language, the distinction is made using hypothetical possible worlds; the extension of 'creature with a heart' is the set of all creatures with a heart in the *actual* world, whilst the intension is the set of creatures with a heart in all *possible* worlds.[17] Equivalently, it would be erroneous to infer from the identity of extensions that

[16] Manders (1987), p.202 [order reversed]
[17] Kearns (2000), pp. 17-18

any creature with a heart will also have a kidney (or two). Thus, the terms have different intensions - i.e., different *meanings*. The hallmark of two terms having different intensions, even when they share a common extension, is the a priori conceivability of one of the terms being true of an object while the other is not - i.e. the hypothetical possibility of such an occurrence.

Traditionally, intension has been seen to have a larger share in philosophically problematic notions, such as reference. It is usually a lot easier to resolve disagreements about extensions than about intensions (through the citing of similar cases, etc). Modern logic, with its concentration on argument *form*, rather than conceptual *content*, is an attempt to render reasoning more perspicuous by eliminating intensions altogether - an argument is valid if and only if it displays one or another (i.e., is in the extension) of the various argument forms, which are selected on the basis of truth-functional connectives. These connectives, by virtue of taking truth-values to truth-values, are likewise extensional objects, as will be explained below. A possible slogan for the extensional approach is 'objects are prior to properties', as properties are defined by the collection of objects of which they hold. An instance if this approach is the standard set-theoretic definition of an n-ary function as a set of ordered n-tuples.

To illustrate how an more intension-sensitive view would look like, consider the model-theoretic definition of what it is for an n-place predicate to apply to n terms:

$\models_A Pt_1...t_n[\overline{s}]$ if and only if[18] $\langle \overline{s}t(_1), ..., \overline{s}(t_n)\rangle \in P^A$

For instance, where the domain $|A| = \mathbf{N}$, P is the '$\leq$' relation, f is the successor function, and c is the constant zero, the open formula

$$\forall x Pyx$$

is not satisfiable (i.e. is not included in the extension of the satisfaction function $\overline{s}$).[19] In the language of model theory, this is because

[18]Hereon abbreviated as 'iff'

[19]Because the formula is open, the free occurrence of y can be regarded as universally quantified, (albeit implicitly).

$\not\models_A Pyx[\overline{s}(x/c)]$

i.e. there is an x, namely c, and a y, $f(c)$, such that

$\langle f(c), c\rangle \notin P^A$

The ordered pair $\langle f(c), c\rangle$, which denotes $\langle 1, 0\rangle$, is not an element of the set P^A, the set which *defines* what the predicate 'P' means in this particular model of the natural numbers. If we were to alter the composition of the set P^A to include $\langle f(c), c\rangle$, then the set would no longer denote the usual '$\leq$' relation. Hence, one alteration to any element of P^A will result in that predicate having a different extension, and thus be a different predicate. The obvious rejoinder is that we choose the extension of the set P^A precisely *because* it will mirror the (intuitive) meaning of '$\leq$', which brings us to our next point.

Regarded from the set- or model-theoretic perspective, functions and predicates are sets or ordered n-tuples of sets, respectively.[20] These sets are given the extensions they have in order to mirror the 'intuitive' meanings of whatever we are setting out to formally describe. Where do these 'intuitive' concepts come from? From an *intensional* point of view, a function can be a method, rule, or operation for turning some things into other things. It is an *action* that can be performed, not a static set of n-tuples. Hence, speaking intensionally, we can regard subtraction in **N** and **Z** as being the same operation (without resort to set theory), as the *method* for obtaining the output is the same: the only difference is that in **Z** some outputs can be obtained which are absent from **N**. An analogous example makes the connection clearer: consider the 'function' of painting objects green. If one is in a rocky locale, then green rocks result from application of the 'function' to the available 'inputs'. However, if one is in a woodland, then green sticks are also attainable. The method, that of painting objects green, is constant. Thus, a recognition of the role of an intensional, informal metalanguage

[20]An interesting possible exception to this is the notion of a 'function-class', a function that is defined by a formula rather than a set of n-tuples. It is called a 'function-class' because it can be defined on all sets, and the universe of all sets is a class (and not a set). If the range of a function includes classes, then it will not be a set. One result of this is that the axiom of replacement in ZF is in fact not an axiom but an axiom *schema*, that is, an infinite set of axioms (see Gowers (ed) 2008, pp.621-2). This may have interesting implications for the relationship between extensions and formulae; unfortunately, such speculations are beyond the scope of this discussion.

can supply us with both the grounds for seeing conceptual continuity between settings while preserving a role for non-trivial accomplishment, as well as the source of the intuitive grounds we appeal to when determining the extensions of the sets which formally define functions and predicates for the purpose of model-theoretic analysis.

Considering the idea that preserving a role for an intensional view of mathematics is fruitful for explaining the existence of both inter-theory continuity of concepts and innovation, it would seem gainful to investigate whether such a stance can shed further light on our more specific task at hand - seeking a source of the 'naturalness' that some settings display with regard to certain theorems over others. In particular, I shall focus on the notion of logical consequence, for it seems that if conceptual continuity of the content of theorems exists between theories, then so must an associated notion of what it is for a theorem or definition to (deductively) imply another.

4 Logical Consequence

For all the philosophical issues attendant to a purely extensional approach to mathematical reasoning, such a stance does afford a very useful *mathematical* technique: a purely formal notion of logical consequence (model theory). The mathematical usefulness of model theory lies in the fact that a mathematical proof can be rendered down into a formal language (usually, but by no means always, first-order logic). This proceeds via the selection of an appropriate formal language and a selection of objects which are to be candidates for the denotations of the variables and constants of the language (this set is the domain; the function which assigns objects to constants and/or variables is the satisfaction function; the domain and the satisfaction function taken together form the model). Once this 'rigourisation' has been carried out, any (legitimate) proof in the original theory can be 'reduced' to a finite number of gap-free steps, all of which are sanctioned by the rules of the appropriate logic. Thus, anyone who wishes to disagree with the result of a given proof will either have to reject one of its premises, or one of its inferences - the latter option intended to be unavailable on pain of irrationality.[21]

Nevertheless, for all this mathematical footwork, it remains that model theory is a mathematical *stand-in* for our 'pre-theoretic' notion of what it is for one thing to follow another. Analogous examples abound; for instance, the continuum is a mathematical model of our pre-theoretic notion of what it is for a straight line to be able to intersect another at *any* point on the line (the notions of 'point' and 'line' are further such instances). Stewart Shapiro notes that in firmly established areas of modern mathematics, model theory comes so close to our pre-theoretic conception of mathematical reasoning that it now serves as the *standard* for logical consequence, rather than a technical stand-in.[22] Shapiro gives the analogy of a dictionary; it serves as the standard of correct spelling and usage, though this is not taken to mean that the dictionary constitutes the origin of those standards. The case is the same with mathematical reasoning, though this is by no means widely acknowledged - the reason for this may be seen as a consequence of the over-emphasis on the formal nature of mathematics which has been present in the philosophy of mathematics over the last hundred years or so.[23]

There are many ways to explicate the pre-theoretic notion of logical consequence, and they may not all be equivalent; such is the lot of pre-theoretic notions. Nevertheless, it does not seem unjustified to take as a working definition the familiar idea of *material* consequence - that is, that

(i) B is a logical consequence of A iff it is impossible for A to be true and B to be false

Hopefully, its inadequacies will be just as illuminating as its successes. Notice that in using the term 'impossible', this is a metaphysical definition (it is to do with the way the world *is*). However, the activity of logic can be seen to have an epistemological, as well as metaphysical dimension; thus (i) can be augmented with the following epistemically flavoured variant

[21]Provided that the method of reduction is also accepted

[22]Shapiro (1998), p.155

[23]cf. Manders (1987), p.196: "The dominant theme in philosophical justification of work and programs in mathematical logic throughout the past century has been that *"logical foundations of mathematics" is a reliability-theoretic enterprise and reliability is the unique necessary and sufficient condition of mathematical knowledge.*" [emphasis original]

(ii) B is a logical consequence of A iff it is irrational to hold that A is true and B is false

So much for the metaphysics/epistemology demarcation. However, before we move on, there is one more consideration that should be noted: the idea of *relevance*. There is a long tradition, stretching back to antiquity, of the idea that one thing can be a logical consequence of another only if the two are somehow related.[24] The concern with relevance arises from the nature of material implication, in that definitions (i) and (ii) do not require any relation of relevance between the premise(s) A and the conclusion B; such inferences, where there is no such connection, are known as 'paradoxes of material implication'. Consider the following (classically) valid inference:

$$P \rightarrow (Q \vee \neg Q)$$

This inference form sanctions the following argument as valid: 'the tree outside my window is green, therefore either the local shop is open or it is not open'. Such inferences can be taken as counterintuitive, for obvious reasons. 'Relevance logics' attempt to avoid such results by (in one instance) restricting implications to holding only between propositions which have one or more variables in common (depending on the logic). While I will not address relevance logics any further, the idea that relevance is implicit in our pre-theoretic notion of logical consequence will return to the forefront of this discussion in the next section.

We have come to see that the formal object-language/informal metalanguage division which is present in the methodology of working mathematicians can be characterised by the extension/intension distinction borrowed from natural language. The formal language of mathematics, whether considered to be first- or higher-order logic, applied via model theory to set theory, is a purely extensional language; all objects and operations are entirely characterised by their relations to other objects and operations. However, holding that in this consists the entirety of the mathematical endeavour presents a dichotomy; either there is no continuity *at all* between different mathematical theories, or there is 'maximal' continuity, in that all mathematical objects from all theories are sets. Both positions do away with the idea that the formulation of a new setting which renders certain theorems more comprehen-

[24] see Shapiro (1998), p.133

sible and/or tractable is a genuine intellectual achievement. But, taking note of the fact that working mathematicians often work within a more informal framework, one which involves conceptual notions construed as intensional, we can avoid the above dichotomous conclusion. This leads us to consider the informal, intensional notion of logical consequence, in particular how it is to be characterised as holding between an old system and a new setting which results in the phenomenon of some theorems seeming more 'natural'. The next section will outline a philosophical position which will accomodate and explain these considerations.

5 Natural Language and the Foundations of Mathematics

From the above considerations, it does not seem unwarranted to assert that when formulating a new setting for the purposes of exploring how new theorems will behave in an altered setting, the mathematical reasoning which occurs takes place primarily in the intensional metalanguage. This conception works for two reasons: one, we look upon the operations which can be performed in the new settings as 'extended' versions of those present in the old system; and two, it preserves the idea that such formulations, when successful, are non-trivial accomplishments.

We would do well to take note of the fact that when a new theory is being formulated and investigated, it is yet to be possessed of the rigorous formalisation into a (usually) first-order language. Thus, the extensional is *yet to be available* as a view of the new theory. It is only once the theory has reached a certain point of maturation that it can be thus codified. However, there are historical cases where one or more of the developments which constitute formalisation have preceded the rest, and thus themselves served as the stimulus for research into a new area.[25] In the words of Kenneth Manders, "...genuine mathematical accomplishment consists primarily in *making* **clear** *by using new concepts.*"[26]

[25]However, there may be historical cases where one or more of the developments which constitute formalisation have preceded the rest, and thus themselves served as the stimulus for research into a new area, which I will not go into here. Nevertheless, it seems that the majority of cases are those which arise from the search for conceptual clarity

[26]Manders (1987), p.193 (emphasis original)

Be that as it may, Manders goes on to observe that "...results also have to be correct to count as making something clear..."[27] But, if the new setting is yet to be mathematically rigourised to the point of constituting a new calculating system, how can these results be *known* to be correct? Hence we observe the return of the reliability condition. It is this question which I will seek to address in this section. To get clear about the kind of conception we need, I will first outline the philosophical position known as 'naïve realism' regarding mathematical objects - also known as 'Platonism'. I will then mention some faults inherent in this position, and go on to formulate an alternative.

The standard contemporary formulation of this position is the conjunction of two claims: (1) that all mathematical theories can be reduced to set theory, and (2) sets exist in a timeless manner which is ontologically independant of the material world (in particular, the beings and doings of mathematicians). For classical mathematics, this includes the claim of the existence of the infinite set. Naïve realism about mathematical objects has many problems. Among these is the well-known issue of epistemic access: if mathematical objects are timeless and do not occupy any spacial location, then it seems that we are incapable of the kind of interaction with them that is required for our gaining knowledge of their properties.[28] Also problematic is the (sometimes unexpected) applicability of mathematics to many real-world phenomena (again, a consequence of mathematical objects' non-physical existence). Due to the already vast existent literature on the subject, I will go no further than to state an alternative to naïve realism which can be described as the 'semantic conception'.[29]

Under the semantic conception, we know mathematical truths because they are truths of meaning; often described as 'true by definition,'[30] we are able to apply them to our physical surroundings because they are constituted by the same phenomena as ordinary language usage. Thus, taking mathematical truth to be a variety of linguistic truth, and mathematical 'objects' as on par with linguistic terms, our epistemic access to, and the empirical applica-

[27]ibid

[28]See Benacerraf (1973) 'Mathematical Truth', *The Journal of Philosophy*, 70:661-669

[29]This view, or a similar one, often goes by the moniker 'conventionalism'; however, this term has considerable historical baggage, which I wish to avoid

[30]And thus analytic. However, as it is not especially pertinent to my case, I will not address the question as to whether mathematical truths are analytic or synthetic (requiring more than knowledge of the meaning of the sub-sentential terms in order for their truth to be determined) or the other philosophical bugbear of whether the analytic/synthetic distinction is ultimately viable.

bility of, mathematics becomes just as problematic as the existence and applicability of the term 'dog'. And while it is undoubtably the case that ordinary natural language has its own philosophical worries, it should soon become clear that not only does the semantic conception of mathematics avoid the pitfalls of naïve realism, it also supplies us with an interesting and novel way of conceiving of the relationship between inter-theory conceptual continuity and its non-trivial attainability, model theory, and our pre-theoretic notion of logical consequence.

Now, the charge most often brought against semantic conceptions of mathematical truth is that they make the discivery/invention of new mathematical ideas 'arbitrary'. By this it is meant that if mathematical truth is in the same conceptual category as linguistic truth, then that truth depends on us; specifically, on what *we accept* as the correct meaning/usage for any given mathematical term. As stated by Shanker:

> The most common criticism levelled against [the semantic conception] is that mathematical truth must be sempiternal and universal: properties that outstrip the reach of conventions, which are rooted to the decisions of a speaker or community. (1987, p.303)

Furthermore:

> The feeling behind [this] criticism is that the [semanticist] contends that a mathematical proposition only expresses either a speaker's or a community's decision to use symbols in a certain way. (ibid, p.304)

Hence, if this truth is dependant on us, then it is inferred that we are 'free' to use the symbols in any way we want; we can make any mathematical statement true or false, almost on a whim. In terms of the example given in §1, the introduction of ideal points is an arbitrary decision designed to simply *rule out* problematic cases, such as when the line projected through a point is parallel to the new plane, resulting projection not being a one-to-one correspondence. Ideal elements allow us to simply 'disregard' such cases. Because this is a decision, and not a discovery, it cannot *tell* us anything. For another example, all that is required for the Goldbach conjecture to be considered true (and thus, actually *be* true) is that all or most mathematicians in the world start considering it to be true. That is, that it becomes a property of all the symbols which denote the natural numbers greater than two and divisible by two that they cannot appear in the formula

$$\forall p_{\in \mathbf{P}} \forall q_{\in \mathbf{P}} \exists n_{\in \mathbf{N}} n \neq p + q$$

where $\mathbf{P}$ is the set of primes. Because, under the semantic conception of mathematical truth, there is no gap between *accepted* truth and *actual* truth, opponents of this consideration take it to be a position which trivialises or disregards the accomplishment inherent in a mathematician's solving a difficult problem. Concurrent is the notion that when formulating a new mathematical system, we are free to make any axiom, rule or object behave in any way we want to, without the constraints normally seen as necessary to have mathematics 'make sense'. I will argue that this is not the case, and in doing so explain why a semantic conception of mathematical truth is able to account for the phenomenon of naturalness.

My argument rests on the following idea: that while it is indeed the case that we are 'free' when constructing new mathematical theories, this does *not* mean that any old way of doing things will suffice. However, I cleave to the semantic conception's notion that mathematical truth does not exist prior to ratification; the truths of (pure) mathematics are not 'out there' to be discovered. We 'make' some mathematical statements true, others false, by regarding them as such. Nevertheless, this 'making' is not done by any force of will or raw desire, but by a process more analogous to being 'convinced'.

Now, it seems plausible to regard most mathematicians as believing that it is possible or desirable for mathematics to one day be organised into a vast interlinking network, where every mathematical truth is a (somewhat) logical outcome of the sum total of all other mathematical truths. Thus it seems we have at least one constraint on the choices to be made when a new system is being constructed: the new system cannot be so strange and bizarre as to be unrecognisable as mathematics to the average mathematician.[31] This, in turn, leads to

[31]It may be argued that there are historical exceptions to this rule; the introduction of 'imaginary' numbers,

further constraints. For one, the rules of derivation (be they explicitly stated, or only used implicitly) must be possessed of some kind of constancy. For instance, 'take propositions A and B; flip a coin; if the result is heads, then B is a logical consequence of A' does not seem acceptable.

Furthermore, and perhaps more importantly, the new theory must be able to 'explain' its presence, usually in terms of theories already existent (which may or may not have served as the initial motivation for the formulation of the new theory in the first place). Thus, analytic geometry 'explains its presence' by being able to offer one singular proof of a situation which may require several synthetic proofs; projective geometry explains itself by its ability to simplify many of the theorems of prior geometries, and so forth. Perhaps one of the most eloquent statements, by way of analogy with musical composition, of this situation is the following:

> A succession of two musical notes is an act of choice; the first causes the second, not in the scientific sense of making it occur necessarily, but in the historical sense of provoking it, of providing it with a motive for occuring. A successful melody is a self-determining history; it is freely what it intends to be, yet is a meaningful whole, not an arbitrary succession of notes. (Auden, 1975, p.465-6[32])

It is in this 'historical sense' that we should regard one mathematical theory as being 'caused' by another. Note that this implies two things - one, mathematical 'objects' are not required beyond being linguistic entities, and thus the naïve realist's epistemic access and applicability worries vanish - and two, the 'explaining itself' that a mathematical theory needs to be seen to do is an ongoing process, one which accompanies the progression toward rigourisation hinted at above. To illustrate my point, I shall give a historical example.

Up until modern times, the axioms of Euclidean geometry were held to be self-evident; this

or maybe Kronecker's opposition to Cantor's explorations of completed infinities. My belief is that it can be counter-argued that (1) these cases will always be in the minority, and (2) the subsequent acceptance of these techniques bears them out. What is important is the attitude of the entire mathematical *community*, not the attitudes of one or two exceptionally vocal and well-placed critics.

[32]Cited in Shanker, 1987, p.338

self-evidence was taken to be known through some kind of 'intuition'. The idea of intuition underpinning the (base) truths of mathematics found its culmination in the philosophy of Immanuel Kant, in his holding that mathematical truths are synthetic a priori (for Kant, all synthetic knowledge is reached via intuition). As is well known, the advent of non-Euclidean geometry had most philosophers of mathematics (and most mathematicians) concluding that it is in fact self-consistency which is the driving force of mathematical truth, and not some kind of psychological faculty or occurrence.[33] These events occurred mostly in the 19$^{\text{th}}$ century; the early 20$^{\text{th}}$ century saw the rise of model theory, the mathematically rigourised counterpart to our pre-theoretic notion of logical consequence.

One pioneering use of model theory was Hilbert's proofs of the independance of the Euclidean axioms from one another in his *Grundlagen der Geometrie*, where his methodology was to give a model for each axiom in which that axiom is true but the others are false. Notice how this squares with the definitions (i) and (ii) of material consequence - Hilbert had given mathematical reasons for why certain mathematical statements are not even pre-theoretic logical consequences of one another; the existence of these models proved that it was metaphysically possible for one axiom to be true while the others are all false, and the mathematical rigour of his method gave compelling reasons to regard it rational to believe that this is the case. Hilbert quotes Kant in that work: "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas".[34] However, Shapiro notes in his analysis of the origins of model theory that "...the plan executed in that work is far from Kantian. In Hilbert's hands, the slogan "passes to concepts and ends with ideas" comes to something like "is *replaced* by logical relations between ideas."" Furthermore, "In Hilbert's writing...the role of intuition is carefully and rigourously limited to *motivation* and *heuristic*."[35] Thus, we can see that the notion of 'intuitive' consequence operant in Kant's conception of the foundations of geometry gets replaced by 'logical consequence' in Hilbert's axiomatisation, where the 'logical' is understood as 'model-theoretic'.

My contention is that it is possible to see Hilbert's *Grundlagen* as a historical, explicit instantiation of what occurs implicitly in most extensions and subsequent rigourisations of

[33]Brouwer and his followers can be regarded as possible exceptions.
[34]*Critique of Pure Reason*, A702/B730. Quoted in Shapiro (1998), p.157
[35]Both ibid; emphasis in the second quotation is mine.

mathematical fields of inquiry. At the outset, intuitive notions of logical consequence and what objects/operations are present in the setting are primarily operant - witness how the propositions of Euclid's *Elements* are things that one is proposed to *do*. Thus, that a theorem follows from the axioms and definitions is ensured by the (possible) construction of a figure, not 'facts of the matter' as to what logically implies what. However, as rigourisation progresses, as it inevitably does in mathematics, the intuitive, intensional notions are supplanted by their extensional counterparts - with (in the present day) the limit case being the embedding of the new system into a set-theoretic metatheory. Such embedding can be seen as the ultimate way for a mathematical theory to 'explain itself' in terms of its antecedants. Nevertheless, the intensional remains, to serve as a 'motivating' and 'heuristic' dimension of the theory - primarily useful teaching and in finding new directions for research. As emphasised above, intuition was not *totally* absent from Hilbert's *Grundlagen*.

As an interesting corollary of these considerations, we can view the gradual transition from the intensional to the extensional as being the source of the feeling of mathematical 'discovery' which makes naïve realism about mathematical objects such an intuitively attractive conception. Under my alternative, it is not the properties of timeless non-physical mathematical objects that are 'discovered', but rather the facts of the matter as to what formal notions best supplant the prior, intuitive ones. Such investigation will be subject to pragmatic concerns; for instance there is usually an implicit 'principle of minimal disturbance' dictating that the structural integrity of the theory is paramount. It was long held that in the postulation of new number systems, the arithmetical properties (uniqueness of sums, distributivity of multiplication over addition, etc) must always remain constant. However, the inclusion in projective geometry of co-ordinate numbers which square to -1 (which results in the inability to form the resulting complex number system into an ordered field) the dictum that *all* properties of numbers be held invariant in *all* number systems is disregarded because of the demonstrate possibility of elegant new theorems. Instances of the fact that such theorems can result from the inclusion of such 'problematic' concepts is what can be rightly said to be 'discovered' in mathematical investigation.

At this juncture, all that remains is to give an analysis of the pre-theoretic notion of logical consequence which is operating in the early, pre-rigourisation phase of the development of a new mathematical theory. As stated above, we should take it to be an intensional conception, that is, more closely aligned with taking the meaning of 'creature with a heart' to be what it *is* for a creature to have a heart, rather than the collection of creatures with hearts. Remember, to intensionally regard a function is to see it as a method or process for turning one object (the input) into another (the output). This is where the notion of relevance enters the picture. In ordinary language, the intuitive idea of what follows from another is (often) intensional, based on content and meaning, and hence concerned with relevance. For instance, a courtroom jury would probably not convict someone of a crime based on an argument from the evidence which included a paradox of material implication - or at least, it seems undesirable that they do so. Thus the existence of a research programme for relevance logics; such research would not be carried out if the paradoxes of material implication were not regarded (by some) as counterintuitive.[36]

This gives us another reason for rejecting the idea that a semantic conception of mathematical truth renders new proofs and theorems arbitrary. If we see the intensional idea of logical consequence as the one at work when new settings are formulated, then the regarding of a new theory as a 'consequence' of the old (perhaps in terms of problems which could not be solved in the old theory, thus motivating the formulation of the new) will mean that the relevance of the new to the old is taken into account, because one is an intensional consequence of the other only if they are somehow related. It should not be possible to point to an instance of a paradox of material implication when it comes to the intensional metatheoretic links between mathematical systems.

6 Conclusion

In §1 I gave an example of the movement to a new setting resulting in conceptual simplification of a mathematical theory. Recopnceptualisation can result in certain theorems appearing more 'natural' when done successfully. §2 began the philosophical analysis of this

[36]It may be asked, if paradoxes of material implication are counterintuitive, why do we codify their possibility into mathematical logic? The answer would seem to be because of the extreme interrelatedness internally displayed by mathematical systems. In such systems, *everything* is relevant.

phenomenon, and brought us to consider the nature of the relationship between theories and their new settings. In an effort to accommodate both conceptual continuity of objects and operations between theories and a regard for mathematical advances as genuine intellectual achievements, we recognised a role for an informal, intensional metalanguage of mathematics in addition to the formal, extensional language used to rigourously express mathematical theorems. This in turn led us to the concept of logical consequence in §3; here also we recognise a role for both the informal (intuitive) and formal (model-theoretic) notions. Having reached these conclusions, we found that a semantic conception of mathematics, where both the intensional and extensional forms of mathematical expression are (metaphysically) fragments of natural language enabled us to both skirt the traditional problems attendant to naïve realism *and* account for our conclusions regarding the naturalness that certain settings display over others. Before I conclude, I will offer an intuitive picture of the conception of mathematical advancement which our conclusions have led us to.

We begin with a fairly mature mathematical theory, either fully formalised or well on its way. Nevertheless, certain conjectures are proving resistant to proof, and/or there are theorems which can be proved but are tediously difficult. A new setting is proposed, for instance one which contains new objects. At this early stage, research is mainly conducted along the lines provided by the conceptual content of the intensional view of the theorems and operations of the old system. However, as this research progresses, the proofs become more and more formal, as the logical investigation of the new setting progresses. When alternative, non-equivalent formal definitions are proposed, their implications are explored and debate ensues over which best fits the intuiive concept being investigated. Eventually, purely formal rules dictating the relationships between the theorems of the new setting are proposed, codifying the results of the more informal proofs. As this progresses, the intuitive aspect plays less and less of a role: the limit case is that of Hilbert's *Grundlagen*, where intuition is relegated to heuristic and motivational purposes only. However, as still being motivational, the intuitive is still present to provide inspiration and hints at possible directions for new research in the future.

Bibliography

Gowers (ed) (2008) *The Princeton Companion to Mathematics* Princeton University Press: Princeton, N.J.

Kearns (2000) *Semantics* MacMillan Press Ltd: London

Mac Lane (1996) 'Structure in Mathematics' *Philosophia Mathematica* Vol. 4 No. 3, pp. 174-183

Manders (1987) 'Logic and Conceptual Relationships in Mathematics' in *Logic Colloquium '85* The Paris Logic Group (eds) Elvesier Science Publishers: North Holland, pp. 193-211

Rosenbaum (1963) *Introduction to Projective Geometry and Modern Algebra* Addison-Wesley: USA

Shanker (1987) *Wittgenstein and the Turning-Point in the Philosophy of Mathematics* State University of New York Press: Albany, N.Y.

Shapiro (1998) 'Logical Consequence: Models and Modality' in *The Philosophy of Mathematics Today* Schirn (ed) Clarendon Press: Oxford, pp.132-156

Wilson (1992) 'Frege: The Royal Road from Geometry' *Noûs* Vol. 26 No.3, pp. 149-180